TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 9, September 1995

STABILITY OF OPTIMAL-ORDER APPROXIMATION BY BIVARIATE SPLINES OVER ARBITRARY TRIANGULATIONS

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ABSTRACT. Let Δ be a triangulation of some polygonal domain in $\mathbb{R}^2$ and $S_k^r(\Delta)$, the space of all bivariate C^r piecewise polynomials of total degree $\leq k$ on Δ. In this paper, we construct a local basis of some subspace of the space $S_k^r(\Delta)$, where $k \geq 3r+2$, that can be used to provide the highest order of approximation, with the property that the approximation constant of this order is independent of the geometry of Δ with the exception of the smallest angle in the partition. This result is obtained by means of a careful choice of locally supported basis functions which, however, require a very technical proof to justify their stability in optimal-order approximation. A new formulation of smoothness conditions for piecewise polynomials in terms of their B-net representations is derived for this purpose.

1. INTRODUCTION

The objective of this paper is to describe the approximation properties of certain bivariate spline spaces over arbitrary triangulations of a polygonal domain in $\mathbb{R}^2$ and to construct the approximants that achieve the highest order of approximation. Let Δ be a 2-dimensional simplicial complex [9, p. 131]. We assume throughout that Δ is pure; that is, each maximal simplex has dimension 2. Then Δ is called a triangulation of a polygonal region in $\mathbb{R}^2$. As usual, for any nonnegative integers k and r, $S_k^r(\Delta)$ denotes the space of all C^r functions which are piecewise polynomials of total degree at most k separated by Δ. The approximation order of the space $S_k^r(\Delta)$ is defined to be the largest integer ρ for which

$$\text{dist}(f, S_k^r(\Delta)) \leq C\,|\Delta|^\rho \tag{1}$$

holds for all sufficiently smooth functions f, where the smallest constant C, called the approximation constant (of optimal-order), depends only on f and the smallest angle in Δ. Also $|\Delta| := \sup_{\tau\in\Delta} \operatorname{diam} \tau$ denotes the mesh size of Δ, and the distance is measured in the supremum norm $\|\ \|$. It is clear

Received by the editors June 22, 1993 and, in revised form, April 4, 1994.

1991 *Mathematics Subject Classification.* Primary 41A25, 41A63; Secondary 41A05, 41A15, 65D07.

Key words and phrases. Bivariate splines, triangulations, B-net representations, approximation order, local bases, stability.

Research of the first and second authors supported by NSF Grant No. DMS 92-06928 and ARO Contract DAAH 04-93-G-0047.

Research of the third author supported in part by NSERC Canada under Grant OGP 121336.

that the approximation order of $S_k^r(\Delta)$ cannot be higher than $k+1$, regardless of r, and is trivially $k+1$ in case $r=0$. On the other hand, it is also well known that for $r \geq 1$ the approximation order from $S_k^r(\Delta)$ not only depends on k and r, but also on the geometric structure of the partition Δ. According to the well-known results in finite element theory (cf. [11]), the full approximation order of $k+1$ is obtained provided that $k \geq 4r+1$. Extension of this property of optimal approximation to $k \geq 3r+2$ is more recent. An abstract proof based on the Hahn-Banach theorem was given by de Boor and Höllig [2]. However, as was already pointed out by de Boor [1] (see also Schumaker [10, p. 547]), the proof given in [2] does not fully support the claim that the approximation constant in (1) depends only on the smallest angle in the triangulation. Although constructive proofs were also given in [5] and [6], yet the behavior of the approximation constants still depends on the measurement of "near-singularity" of Δ; i.e., the constant becomes large for near-singular vertices. Observe that when Δ is refined so that $|\Delta| \to 0$ in (1), the standard refinement algorithms are mainly concerned with the smallest angle in the partitions, but not with the "near-singularity" of such refinement. Therefore, it is important to give an approximation scheme in order to show that the spline space $S_k^r(\Delta)$, $k \geq 3r+2$, admits optimal approximation order of $k+1$ in such a way that the approximation constant C in (1) does not depend on the geometry (such as near-singularity), with the exception of the smallest angle in Δ.

The main purpose of this paper is to construct a stable local basis of the super spline subspace $S_k^{r,\mu}(\Delta)$ of $S_k^r(\Delta)$, where $k \geq 3r+2$ and $\mu = \lfloor \frac{3r+1}{2} \rfloor$ (see [4, p. 73] and [10]), and to show that the full order of approximation can be achieved via a quasi-interpolation scheme using this basis, and that the approximation constant C in (1) of this optimal order depends only on the smallest angle in the triangulation Δ. Unlike the techniques introduced in [2] (see also [1]), which are based on determining the smoothness conditions in terms of the domain points on two triangles that share a common edge to "disentangle the rings" of smoothness conditions, our approach is to inductively determine the smoothness conditions on the rings of the domain points of all vertices; that is, we determine the smoothness conditions in terms of the points on all of the triangles attached to a common vertex.

This paper is organized as follows. In Section 2, in order to facilitate our procedure of constructing a stable super spline basis, we give a new formulation of the smoothness conditions in terms of the B-net representations. In Section 3, we demonstrate how to choose a minimum determining set and provide an explicit scheme of approximation from $S_k^{r,\mu}(\Delta)$ that attains the optimal approximation order. Finally, in Section 4, we will give an explicit scheme for constructing some stable local basis of $S_k^{r,\mu}(\Delta)$.

2. Preliminaries

Throughout this paper, we will always assume, without loss of generality, that Δ is connected. For a vertex v of Δ, we denote by $\overline{St}(v)$ the closed star of vertex v in Δ [9, p. 135]; i.e., the cell formed by all the triangles in Δ with v as a common vertex. If $\overline{St}(v)\backslash\{v\}$ is connected for every vertex v of Δ, then Δ is called strongly connected. If Δ is strongly connected, then each boundary vertex has exactly two boundary edges attached to it. For simplicity, we will always assume that Δ is strongly connected, though our discussion is also valid otherwise.

Let $\tau = [u, v, w]$ be a triangle with vertices u, v and w. For any $x \in \mathbb{R}^2$, denote by $\xi(x) = (\xi_u, \xi_v, \xi_w)$ the barycentric coordinates of x with respect to τ; that is,

$$x = \xi_u u + \xi_v v + \xi_w w, \quad \xi_u + \xi_v + \xi_w = 1.$$

For $\alpha = (\alpha_u, \alpha_v, \alpha_w) \in \mathbb{Z}_+^3$, the Bernstein-Bézier polynomial $B_{\alpha,\tau}$ is defined by

$$B_{\alpha,\tau}(x) = \binom{|\alpha|}{\alpha} \xi_u^{\alpha_u} \xi_v^{\alpha_v} \xi_w^{\alpha_w},$$

where $|\alpha| = \alpha_u + \alpha_v + \alpha_w$ and $\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha_u! \alpha_v! \alpha_w!}$. Moreover, we define the points $x_{\alpha,\tau}$ on τ to be $(\alpha_u u + \alpha_v v + \alpha_w w)/|\alpha|$. It is well-known that any $p \in \pi_k$, the space of all polynomials of total degree $\le k$, can be written in a unique way as

$$p = \sum_{|\alpha|=k} b_{\alpha,\tau} B_{\alpha,\tau}.$$

This gives rise to a mapping b: $x_{\alpha,\tau} \mapsto b_{\alpha,\tau}$, $|\alpha| = k$, and this mapping is called the B-net representation of p with respect to τ.

Let X be the collection $\{x_{\alpha,\tau}: \tau \in \Delta, |\alpha| = k\}$. To any $f \in S_k^0(\Delta)$ there corresponds a unique mapping b_f from X to $\mathbb{R}$ such that on each $\tau \in \Delta$,

$$f|_\tau = \sum_{|\alpha|=k} b_f(x_{\alpha,\tau}) B_{\alpha,\tau}.$$

This b_f is called the B-net representation of f with respect to Δ.

In our investigation, it is essential to represent C^r-smoothness conditions of spline functions in terms of B-net representations. Suppose that a spline function f is defined over two triangles, say $\tau = [u, v, w]$ and $\tilde{\tau} = [u, v, \tilde{w}]$, with a common edge $[u, v]$. Let S, S_u, S_v and S_w denote the oriented areas of the triangles τ, $[\tilde{w}, v, w]$, $[u, \tilde{w}, w]$, and $\tilde{\tau}$, respectively. For instance, if u is the origin of $\mathbb{R}^2$, $v = (v_1, v_2)$, $w = (w_1, w_2)$ and $\tilde{w} = (\tilde{w}_1, \tilde{w}_2)$, then

$$S = \frac{1}{2}(v_1 w_2 - v_2 w_1) \tag{2}$$

and

$$S_v = \frac{1}{2}(\tilde{w}_1 w_2 - \tilde{w}_2 w_1), \quad S_w = \frac{1}{2}(v_1 \tilde{w}_2 - v_2 \tilde{w}_1). \tag{3}$$

The following lemma describes C^r-smoothness conditions on a spline function f in terms of its B-net representation (cf. [8]).

Lemma 1. *Suppose that a function f is defined on $\tau \cup \tilde{\tau}$ by*

$$f|_\tau = \sum_{|\alpha|=k} b(x_{\alpha,\tau}) B_{\alpha,\tau};$$

$$f|_{\tilde{\tau}} = \sum_{|\alpha|=k} b(x_{\alpha,\tilde{\tau}}) B_{\alpha,\tilde{\tau}}.$$

Then $f \in C^r(\tau \cup \tilde{\tau})$ *if and only if for all nonnegative integers* $\ell \le r$ *and* $\gamma = (\gamma_u, \gamma_v, 0) \in \mathbb{Z}_+^3$ *with* $|\gamma| = k - \ell$,

$$(4) \qquad b(x_{\gamma+\ell e^3}, \tilde{\tau}) = \sum_{|\beta|=\ell} b(x_{\gamma+\beta}, \tau) \binom{\ell}{\beta} \left(\frac{S_u}{S}\right)^{\beta_u} \left(\frac{S_v}{S}\right)^{\beta_v} \left(\frac{S_w}{S}\right)^{\beta_w},$$

where $\beta = (\beta_u, \beta_v, \beta_w) \in \mathbb{Z}_+^3$ *and* $e^3 = (0, 0, 1)$.

We remark that the quantities $\frac{S_u}{S}$, $\frac{S_v}{S}$, and $\frac{S_w}{S}$ are all bounded by some constant which depends only on the smallest angle in the partition Δ. Hence, as we will see, the approximation constant for optimal-order approximation has to depend on this smallest angle.

For later reference, we need another form of the smoothness conditions which plays an important role to prove the stability of the local basis. For $\alpha = (\alpha_u, \alpha_v, \alpha_w) \in \mathbb{Z}_+^3$ with $|\alpha| = k$, let

$$(5) \qquad C_{\alpha,\tau} := \sum_{\beta_v=0}^{\alpha_v} \sum_{\beta_w=0}^{\alpha_w} (-1)^{\alpha_v - \beta_v + \alpha_w - \beta_w} \binom{\alpha_v}{\beta_v} \binom{\alpha_w}{\beta_w} b\big(x_{(k-\beta_v-\beta_w, \beta_v, \beta_w)}, \tau\big).$$

Then we have the following result.

Lemma 2. *A function* $s \in S_k^0(\tau \cup \tilde{\tau})$ *is in* C^r *if and only if the corresponding terms* $\{C_{\alpha,\tau}\}$ *and* $\{C_{\alpha,\tilde{\tau}}\}$ *satisfy the conditions*:

$$(6) \qquad C_{\alpha,\tilde{\tau}} = \sum_{\ell=0}^{\alpha_w} C_{(\alpha_u, \alpha_v+\ell, \alpha_w-\ell), \tau} \binom{\alpha_w}{\ell} \frac{S_v^\ell S_w^{\alpha_w-\ell}}{S^{\alpha_w}},$$

for $1 \le \alpha_w \le r$ *where* $\alpha = (\alpha_u, \alpha_v, \alpha_w) \in \mathbb{Z}_+^3$ *with* $|\alpha| = k$.

Proof. Without loss of generality, we may assume that u is the origin of $\mathbb{R}^2$. Let $\tau = [u, v, w]$, $\tilde{\tau} = [u, v, \tilde{w}]$, and consider an $s \in S_k^0(\tau \cup \tilde{\tau})$ which agrees with some $p \in \pi_k$ on τ and $\tilde{p} \in \pi_k$ on $\tilde{\tau}$. For $0 \le m \le k$, let p_m and $\tilde{p}_m$ be the homogeneous components of degree m of p and $\tilde{p}$, respectively. Also, let s_m be the corresponding piecewise polynomial function which agrees with p_m on τ and $\tilde{p}_m$ on $\tilde{\tau}$. Clearly, $s_m \in S_k^0(\tau \cup \tilde{\tau})$, $m = 0, 1, \dots, k$. Moreover, since we may assume that the mesh line $[u, v]$ is on the x-axis, it is not difficult to see that s is in C^r if and only if each s_m is in C^r, $m = 0, 1, \dots, k$. Note that

$$p(x) = \sum_{\nu_v+\nu_w \le k} b(x_\nu, \tau) \frac{k!}{(k-\nu_v-\nu_w)!\nu_v!\nu_w!} (1-\xi_v-\xi_w)^{k-\nu_v-\nu_w} \xi_v^{\nu_v} \xi_w^{\nu_w},$$

where $(1-\xi_v-\xi_w, \xi_v, \xi_w)$ is the barycentric coordinate of x with respect to the triangle $\tau = [u, v, w]$. By the multinomial theorem, we have

$$\begin{aligned}(1-\xi_v-\xi_w)^{k-\nu_v-\nu_w} \\ = \sum_{\delta_v+\delta_w \le k-\nu_v-\nu_w} \frac{(k-\nu_v-\nu_w)!}{(k-\nu_v-\nu_w-\delta_v-\delta_w)!\delta_v!\delta_w!} (-1)^{\delta_v+\delta_w} \xi_v^{\delta_v} \xi_w^{\delta_w}.\end{aligned}$$

Therefore, by setting $\delta_v + \nu_v$ and $\delta_w + \nu_w$ to be α_v and α_w, respectively, we have

$$p(x) = \sum_{\alpha_v+\alpha_w\le k} \sum_{\substack{\nu_v\le\alpha_v \\ \nu_w\le\alpha_w}} b(x_{\nu,\tau}) \frac{k!}{(k-\alpha_v-\alpha_w)!\alpha_v!\alpha_w!} \cdot \binom{\alpha_v}{\nu_v}\binom{\alpha_w}{\nu_w}(-1)^{\alpha_v-\nu_v+\alpha_w-\nu_w}\xi_v^{\alpha_v}\xi_w^{\alpha_w}.$$

Recall that $u = (0, 0)$. So, writing $v = (v_1, v_2)$, $w = (w_1, w_2)$, and $x = (x_1, x_2)$, we see that

$$\xi_v = \frac{\text{area}[u, x, w]}{\text{area}[u, v, w]} = \frac{x_1w_2 - x_2w_1}{v_1w_2 - v_2w_1}$$

and

$$\xi_w = \frac{\text{area}[u, v, x]}{\text{area}[u, v, w]} = \frac{v_1x_2 - v_2x_1}{v_1w_2 - v_2w_1}$$

are homogeneous linear functions of x; hence, we have

$$p_m(x) = \sum_{\alpha_v+\alpha_w=m} \sum_{\substack{\nu_v\le\alpha_v \\ \nu_w\le\alpha_w}} b(x_{\nu,\tau}) \frac{k!}{(k-m)!\alpha_v!\alpha_w!} \cdot \binom{\alpha_v}{\nu_v}\binom{\alpha_w}{\nu_w}(-1)^{\alpha_v-\nu_v+\alpha_w-\nu_w}\xi_v^{\alpha_v}\xi_w^{\alpha_w}.$$

Taking (5) into account, we deduce that

$$p_m(x) = \sum_{\alpha_v+\alpha_w=m} \frac{k!}{(k-m)!\alpha_v!\alpha_w!} C_{\alpha,\tau}\xi_v^{\alpha_v}\xi_w^{\alpha_w}. \tag{7}$$

Similarly, we have

$$\tilde{p}_m(x) = \sum_{\alpha_v+\alpha_{\tilde{w}}=m} \frac{k!}{(k-m)!\alpha_v!\alpha_{\tilde{w}}!} C_{\alpha,\tilde{\tau}}\tilde{\xi}_v^{\alpha_v}\tilde{\xi}_{\tilde{w}}^{\alpha_{\tilde{w}}}, \tag{8}$$

where $(1-\tilde{\xi}_v-\tilde{\xi}_{\tilde{w}}, \tilde{\xi}_v, \tilde{\xi}_{\tilde{w}})$ is the barycentric coordinate of x with respect to $[u, v, \tilde{w}]$.

Let us now express ξ_v and ξ_w in terms of $\tilde{\xi}_v$ and $\tilde{\xi}_{\tilde{w}}$. Suppose $\tilde{w} = (\tilde{w}_1, \tilde{w}_2)$. Then we have

$$\tilde{\xi}_v = \frac{x_1\tilde{w}_2 - x_2\tilde{w}_1}{v_1\tilde{w}_2 - v_2\tilde{w}_1}, \quad \tilde{\xi}_{\tilde{w}} = \frac{v_1x_2 - v_2x_1}{v_1\tilde{w}_2 - v_2\tilde{w}_1}.$$

These two equalities together with (2) and (3) yield

$$\xi_w = \frac{S_w}{S}\tilde{\xi}_{\tilde{w}}$$

and

$$\xi_v = \tilde{\xi}_v + \frac{S_v}{S}\tilde{\xi}_{\tilde{w}}.$$

Finally, replacing ξ_v by $\tilde{\xi}_v + \frac{S_v}{S}\tilde{\xi}_{\tilde{w}}$ and ξ_w by $\frac{S_w}{S}\tilde{\xi}_{\tilde{w}}$ in (7), we obtain

$$\begin{aligned} p_m(x) &= \sum_{\alpha_v+\alpha_w=m} \frac{k!}{(k-m)!\alpha_v!\alpha_w!} C_{\alpha,\tau} \left(\tilde{\xi}_v + \frac{S_v}{S}\tilde{\xi}_{\tilde{w}}\right)^{\alpha_v} \left(\frac{S_w}{S}\tilde{\xi}_{\tilde{w}}\right)^{\alpha_w} \\ &= \sum_{\alpha_v+\alpha_w=m} \sum_{\ell=0}^{\alpha_v} \frac{k!}{(k-m)!\alpha_v!\alpha_w!} \frac{\alpha_v!}{\ell!(\alpha_v-\ell)!} C_{\alpha,\tau} \\ &\qquad \cdot \left(\frac{S_v}{S}\right)^{\alpha_v-\ell} \left(\frac{S_w}{S}\right)^{\alpha_w} \tilde{\xi}_v^{\ell}\tilde{\xi}_{\tilde{w}}^{\alpha_v-\ell+\alpha_w}. \end{aligned}$$

Let $\beta_{\tilde{w}} = \alpha_v + \alpha_w - \ell$, $\beta_v = \ell$ and $q = \alpha_v - \ell$. Then we have $\alpha_w = \beta_{\tilde{w}} - q$, and

$$p_m(x) = \sum_{\beta_v+\beta_{\tilde{w}}=m} \frac{k!}{(k-m)!\beta_v!\beta_{\tilde{w}}!} \sum_{q=0}^{\beta_{\tilde{w}}} \binom{\beta_{\tilde{w}}}{q} C_{(k-m,\,\beta_v+q,\,\beta_{\tilde{w}}-q),\tau} \frac{S_v^q S_w^{\beta_{\tilde{w}}-q}}{S^{\beta_{\tilde{w}}}} \tilde{\xi}_v^{\beta_v}\tilde{\xi}_{\tilde{w}}^{\beta_{\tilde{w}}}.$$

Comparing this with the expression for $\tilde{p}_m(x)$ in (8), we conclude that $s_m \in C^r$ if and only if

$$C_{\beta,\tilde{\tau}} = \sum_{q=0}^{\beta_{\tilde{w}}} \binom{\beta_{\tilde{w}}}{q} C_{(\beta_u,\,\beta_v+q,\,\beta_{\tilde{w}}-q),\tau} \frac{S_v^q S_w^{\beta_{\tilde{w}}-q}}{S^{\beta_{\tilde{w}}}}$$

for $\beta = (\beta_u, \beta_v, \beta_{\tilde{w}}) \in \mathbb{Z}_+^3$ with $1 \le \beta_{\tilde{w}} \le r$ and $\beta_v + \beta_{\tilde{w}} = m$. This completes the proof of the lemma.

3. Main results

To investigate the approximation properties of bivariate spline spaces, it is convenient to introduce the notion of super splines. Given a triangulation Δ and nonnegative integers k, r and μ with $k \ge \mu \ge r$, a super spline is a piecewise polynomial of degree at most k on Δ which is C^r across each edge and C^μ around each vertex. Let $S_k^{r,\mu}(\Delta)$ be the space of all such splines. Then $S_k^{r,\mu}(\Delta)$ is a subspace of $S_k^r(\Delta)$. In this section, we describe an explicit quasi-interpolation scheme and prove that the super spline space $S_k^{r,\mu}(\Delta)$, $\mu = \lfloor \frac{3r+1}{2} \rfloor$, $k \ge 3r+2$, admits the optimal approximation order of $k+1$ with the approximation constant dependent only on the smallest angle in the partition Δ.

Let us introduce a natural pairing

$$\langle \lambda, b \rangle := \sum_{x\in X} \lambda(x)b(x), \quad \lambda, b \in \mathbb{R}^X,$$

on $\mathbb{R}^X$. Now choose and fix an orientation for each interior edge of Δ. Let $e = [u, v]$ be an oriented interior edge with two triangles $[u, v, w]$ and $[u, v, \tilde{w}]$ attached to it. If the orientation of e is from u to v, then we assume that the points u, v and w are ordered in the counterclockwise direction. In this case, we say that the orientation of the triangle τ agrees with the orientation of the edge e. Let $\alpha = (\alpha_u, \alpha_v, \alpha_{\tilde{w}}) \in \mathbb{Z}_+^3$ with $|\alpha| = k$ and $\alpha_{\tilde{w}} \ge 1$. Bearing

Lemma 1 in mind, we define a function $\lambda_{e,\alpha}$ on X as follows:

$$\lambda_{e,\alpha}(x) := \begin{cases} 1, & \text{if } x = x_{\alpha,\tilde{\tau}}; \\ -\binom{\alpha_{\tilde{w}}}{\beta}\frac{S_u^{\beta_u}S_v^{\beta_v}S_w^{\beta_w}}{S^{\alpha_{\tilde{w}}}}, & \text{if } x = x_{(\alpha_u,\alpha_v,0)+\beta,\tau} \\ & \text{for } \beta \in \mathbb{Z}_+^3 \text{ with } |\beta| = \alpha_{\tilde{w}}; \\ 0, & \text{elsewhere.} \end{cases} \tag{9}$$

The points $x_{\alpha,\tau}$ and $x_{\alpha,\tilde{\tau}}$ will be called the tips of $\lambda_{e,\alpha}$.

In the sequel, we always assume that $k \geq 3r+2$ and consider $\mu := \lfloor \frac{3r+1}{2} \rfloor$. For a vertex u and an oriented interior edge e attached to u, we consider the collections $\Lambda_{u,e}^n$ defined by

$$\Lambda_{u,e}^n = \{\lambda_{e,\alpha}\colon \alpha_u = k-n\}, \quad n = 1, 2, \ldots, \mu, \tag{10}$$

and

$$\Lambda_{u,e}^n := \{\lambda_{e,\alpha}\colon \alpha_u = k-n;\ \alpha_v,\ \alpha_{\tilde{w}} \leq r\}, \quad n = \mu+1, \ldots, 2r. \tag{11}$$

Furthermore, let

$$\Lambda_u^n := \bigcup_{e \ni u} \Lambda_{u,e}^n, \qquad n = 1, 2, \ldots, 2r, \tag{12}$$
$$\Lambda_u := \bigcup_{n=1}^{2r} \Lambda_u^n,$$

and for an oriented interior edge e, let

$$\Lambda_e := \{\lambda_{e,\alpha}\colon 1 \leq \alpha_{\tilde{w}} \leq r < \alpha_u,\ \alpha_v < k-\mu\}. \tag{13}$$

Finally, let

$$\Lambda := \left(\bigcup_{u \in V} \Lambda_u\right) \cup \left(\bigcup_{e \in E} \Lambda_e\right), \tag{14}$$

where V and E denote the collections of vertices and oriented interior edges of Δ, respectively. By Lemma 1, we see that $f \in S_k^{r,\mu}(\Delta)$ if and only if its B-net representation b_f satisfies

$$\langle \lambda, b_f \rangle = 0, \qquad \lambda \in \Lambda.$$

A subset Y of X is called a determining set for the super spline space $S_k^{r,\mu}(\Delta)$, if the linear mapping $f \mapsto b_f|_Y$ defined on $S_k^{r,\mu}(\Delta)$ is one-to-one. Our goal is to find a minimum determining set for this super spline space.

An interior vertex u is said to be singular, if there are exactly four edges attached to it and these edges lie on two straight lines. Otherwise, u is called nonsingular. In particular, a boundary vertex is regarded as nonsingular.

For a vertex u and a triangle $\tau = [u, v, w]$ attached to u, let

$$X_{u,\tau}^n := \{x_{\alpha,\tau}\colon \alpha_u = k-n\}, \quad n = 0, 1, \ldots, \mu; \tag{15}$$
$$X_u^n := \bigcup_{\tau \ni u} X_{u,\tau}^n, \quad n = 0, 1, \ldots, \mu.$$

We associate with each vertex u a triangle τ attached to u and define

$$Y_u^n := X_{u,\tau}^n, \quad n = 0, 1, \ldots, \mu. \tag{16}$$

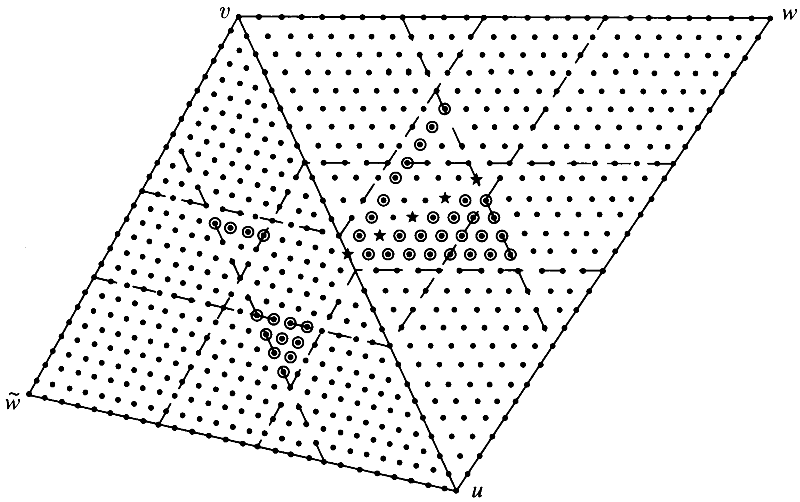

FIGURE 1. The points "$\odot$" in $X^n_{u,e}$, $n = \mu+1, \ldots, 2r$, and $X^n_{v,e}$, $n = \mu + 1$, and "$\star$" in Y_e for nonsingular vertex u $(k = 26, r = 8, \mu = 12)$.

Let e be any oriented interior edge with a given u and some v as two of its vertices. Also let τ and $\tilde{\tau}$ be the two triangles attached to e, such that the orientation of τ agrees with that of e; moreover, denote by w and $\tilde{w}$ the third vertices of τ and $\tilde{\tau}$, respectively. For $n = \mu + 1, \cdots, 2r$, if u is a nonsingular vertex, we define $X^n_{u,e}$ to be the union of the two sets

$$\{x_{\alpha,\tilde{\tau}}\colon \alpha_u = k - n;\ n - r \le \alpha_{\tilde{w}} \le r\}$$

and

$$\{x_{\alpha,\tau}\colon \alpha_u = k - n;\ 2n - 3r - 1 \le \alpha_w \le r\}$$

(see Figure 1). If u is a singular vertex and $\tau = [u, v, w]$ is a triangle attached to u, we define

$$X^n_{u,\tau} := \{x_{\alpha,\tau} : \alpha_u = k - n;\ n - r \le \alpha_w \le r\}, \quad n = \mu + 1, \ldots, 2r,$$

(see Figure 2). If e is an oriented edge attached to a singular vertex u, we set

$$X^n_{u,e} := X^n_{u,\tau} \cup X^n_{u,\tilde{\tau}}, \quad n = \mu + 1, \ldots, 2r,$$

where τ and $\tilde{\tau}$ are the two triangles with common edge e; also, set

$$Y^n_u := X^n_{u,\tau}, \quad n = \mu + 1, \ldots, 2r,$$

where τ is an arbitrarily chosen triangle attached to u. For any vertex u, singular or otherwise, we define

$$X^n_u := \bigcup_{e \ni u} X^n_{u,e}, \quad n = \mu + 1, \ldots, 2r. \tag{17}$$

Furthermore, we associate with each oriented interior edge e three sets

$$X^+_e := \{x_{\alpha,\tau} :\ 0 \le \alpha_w \le r < \alpha_u, \alpha_v < k - \mu\},$$

$$X^-_e := \{x_{\alpha,\tilde{\tau}} :\ 1 \le \alpha_{\tilde{w}} \le r < \alpha_u, \alpha_v < k - \mu\},$$

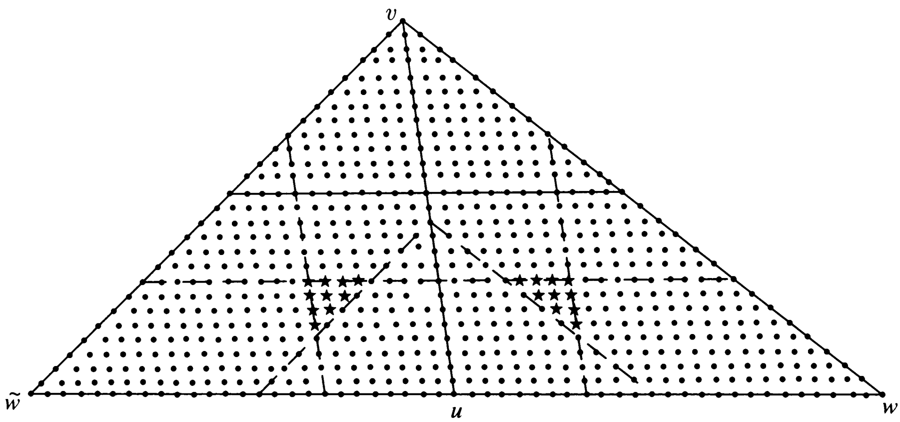

FIGURE 2. The points in $X^n_{u,e}$, $n = \mu+1, \ldots, 2r$, for a singular vertex u ($k = 26$, $r = 8$, $\mu = 12$).

and

$$Y_e := X_e^+ \setminus \bigcup_{n=\mu+1}^{2r} (X^n_{u,e} \cup X^n_{v,e}). \tag{18}$$

Finally, for each triangle τ, we define

$$X_\tau := \{x_{\alpha,\tau} : \alpha_u, \alpha_v, \alpha_w > r\}.$$

From the preceding construction we see that X is the disjoint union

$$\left(\bigcup_{n=0}^{2r} \bigcup_{u\in V} X^n_u\right) \cup \left(\bigcup_{e\in E} (Y_e \cup X_e^-)\right) \cup \left(\bigcup_{\tau\in\Gamma} X_\tau\right),$$

where Γ denotes the collection of all triangles in Δ.

Suppose now that u is a nonsingular vertex. Then for each integer n between $\mu+1$ and $2r$, we choose a subset Z^n_u of X^n_u such that the cardinality $\#Z^n_u$ of Z^n_u is equal to $\#\Lambda^n_u$, and

$$|\det(\lambda(x))_{\lambda\in\Lambda^n_u, x\in Z^n_u}| \geq |\det(\lambda(x))_{\lambda\in\Lambda^n_u, x\in Z}| \tag{19}$$

for any subset Z of X^n_u with $\#Z = \#\Lambda^n_u$. It is known that the matrix $(\lambda(x))_{\lambda\in\Lambda^n_u, x\in X^n_u}$ has full (row) rank (see [2, Proposition 6] and [7]); hence

$$\det(\lambda(x))_{\lambda\in\Lambda^n_u, x\in Z^n_u} \neq 0,$$

and we will write

$$Y^n_u := X^n_u \setminus Z^n_u, \quad n = \mu+1, \ldots, 2r. \tag{20}$$

For each triangle $\tau \in \Delta$, we define

$$Y_\tau := X_\tau.$$

Finally, we set

$$Y_v := \bigcup_{n=0}^{2r} Y^n_v,$$

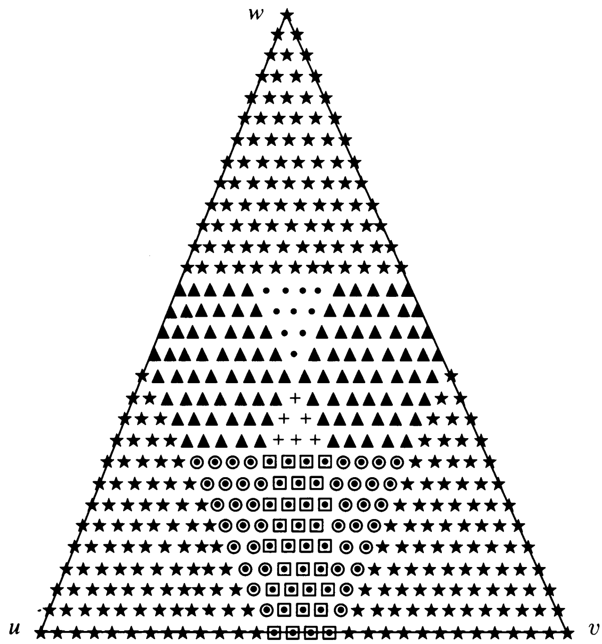

FIGURE 3. The classification of point set X on a triangle.

and

$$Y := \left(\bigcup_{v\in V} Y_v\right) \cup \left(\bigcup_{e\in E} Y_e\right) \cup \left(\bigcup_{\tau\in\Gamma} Y_\tau\right). \tag{21}$$

Then from the following theorem, we see that Y is a minimum determining set for $S_k^{r,\mu}(\Delta)$.

Theorem 1. *For each b: $Y \mapsto \mathbb{R}$, there exists a unique $f \in S_k^{r,\mu}(\Delta)$ such that the B-net representation b_f of f satisfies*

$$b_f|_Y = b.$$

Proof. Let $\Lambda^\perp := \{b \in \mathbb{R}^X\colon \langle\lambda, b\rangle = 0,\ \lambda \in \Lambda\}$, where Λ is given by (14). Then $f \in S_k^{r,\mu}(\Delta)$ if and only if $b_f \in \Lambda^\perp$; hence, it suffices to show that for a given mapping $b : Y \mapsto \mathbb{R}$, there exists a unique $\widehat{b} \in \Lambda^\perp$ such that $\widehat{b}|_Y = b$.

We shall first extend b to $\bigcup_{u\in V} X_u^n$ for $n = 0, 1, \dots, \mu$. For each $u \in V$, (16) tells us that $Y_u^n = X_{u,\tau}^n$ for $n = 0, 1, \dots, \mu$, where τ is a triangle attached to u. On the other hand, there exists a polynomial $p_u \in \pi_k$ such that its B-net representation b_{p_u} on Δ satisfies $b_{p_u}(x) = b(x)$ for all $x \in \bigcup_{n=0}^{\mu} Y_u^n$. We extend b to $\bigcup_{n=0}^{\mu} X_u^n$ by setting $\widehat{b}(x) := b_{p_u}(x)$ for every $x \in \bigcup_{n=0}^{\mu} X_u^n$. Evidently, $\langle\lambda, \widehat{b}\rangle = 0$ for all $\lambda \in \bigcup_{n=1}^{\mu} \Lambda_u^n$.

Next, we extend b to $\bigcup_{u\in V} X_u^n$ for $n = \mu + 1, \dots, 2r$. This is done inductively on n as follows. Let n be given with $\mu + 1 \le n \le 2r$. Suppose that $\widehat{b}(x)$ has been determined for $x \in \bigcup_{j=0}^{n-1} \bigcup_{u\in V} X_u^j$ in such a way that $\langle\lambda, \widehat{b}\rangle = 0$ for all $\lambda \in \bigcup_{j=1}^{n-1} \bigcup_{u\in V} \Lambda_u^j$. We wish to determine the values of $\widehat{b}$ on $\bigcup_{u\in V} X_u^n$

such that for every $u \in V$,

$$\sum_{x \in X} \lambda(x)\widehat{b}(x) = 0, \quad \lambda \in \Lambda_u^n.$$

We claim that the value $\widehat{b}(x)$ has been determined whenever $x \in X \setminus X_u^n$ and $\lambda(x) \neq 0$ for some $\lambda \in \Lambda_u^n$. To establish this claim, we consider $\lambda = \lambda_{e,\alpha}$, where e is an edge attached to u and $\alpha = (\alpha_u, \alpha_v, \alpha_w)$ with $\alpha_u = k - n$. Without loss of generality, we assume that $e = [u, v]$ is an oriented interior edge and both of u and v are nonsingular, for otherwise the proof is analogous. Let $\tau = [u, v, w]$ and $\tilde{\tau} = [u, v, \tilde{w}]$ be the two triangles with common edge e. If $\lambda(x) \neq 0$ and $x \notin X_u^n$, then $x = x_{\beta,\tau}$ for some $\beta \in \mathbb{Z}_+^3$ with $|\beta| = k$, $\beta_u \geq \alpha_u$, $\beta_v \geq \alpha_v$. Thus, we have $\beta_u = k - m$ for some $m \leq n$. Since $\widehat{b}(x)$ has been determined for $x \in \bigcup_{j=0}^{n-1} \bigcup_{u \in V} X_u^j$, we may assume that $x \notin \bigcup_{j=0}^{n-1} \bigcup_{u \in V} X_u^j$. It follows that $\mu + 1 \leq m \leq 2r$. By the definition of X_u^m, we have $\beta_w < 2m - 3r - 1$. Consequently, $x_{\beta,\tau} \notin X_v^p$ for any $p \geq m$. This shows that

$$x_{\beta,\tau} \notin \bigcup_{n=\mu+1}^{2r} (X_{u,e}^n \cup X_{v,e}^n).$$

By the definition of Y_e, we have $x = x_{\beta,\tau} \in Y_e$, and therefore $\widehat{b}(x) = b(x)$ is already determined. This verifies our claim.

Suppose now that u is a nonsingular vertex. Remember that X_u^n is the disjoint union of Y_u^n and Z_n^n and $\hat{b}(x) = b(x)$ for $x \in Y_u^n$. Moreover,

$$\det\big(\lambda(x)\big)_{\lambda \in \Lambda_u^n, x \in Z_u^n} \neq 0.$$

Thus, the values of $\widehat{b}$ on Z_u^n can be determined by solving the system

$$\sum_{x \in Z_u^n} \lambda(x)\hat{b}(x) = - \sum_{x \in X \setminus Z_u^n} \lambda(x)\hat{b}(x), \quad \lambda \in \Lambda_u^n$$

of linear equations.

It remains to deal with the case where u is a singular vertex. Suppose that τ_1, τ_2, τ_3 and τ_4 are the four triangles attached to u and arranged in a consecutive way. We may assume that $Y_u^n = X_{u,\tau_1}^n$, $n = \mu+1, \ldots, 2r$. Let e_j, $j = 1, 2, 3, 4$, be the common edge of τ_j and τ_{j+1} with $\tau_5 := \tau_1$. We have $\widehat{b}(x) = b(x)$ for $x \in Y_u^n = X_{u,\tau_1}^n$. Note that the matrix $\big(\lambda(x)\big)_{\lambda \in \Lambda_{u,e_j}^n, x \in X_{u,\tau_{j+1}}^n}$ is a nonsingular diagonal matrix if the λ's and the x's are arranged in an appropriate way. Thus, we can determine the values of $\widehat{b}$ on $X_{u,\tau_{j+1}}^n$, $j = 1, 2, 3$, by solving the system

$$\sum_{x \in X_{u,\tau_{j+1}}^n} \lambda(x)\hat{b}(x) = - \sum_{x \in X \setminus X_{u,\tau_{j+1}}^n} \lambda(x)\hat{b}(x), \quad \lambda \in \Lambda_{u,e_j}^n$$

of linear equations.

It is known (see, e.g., [7]) that the function $\hat{b}$ so obtained also satisfies

$$\sum_{x \in X} \lambda(x)\hat{b}(x) = 0 \quad \text{for all } \lambda \in \Lambda_{u,e_4}^n.$$

Finally, we extend b to all the remaining points X; i.e., to the points in $\bigcup_{e\in E} X_e^-$. This can be easily done by applying the smoothness conditions (4) across each interior edge.

To summarize, we have constructed a function $\widehat{b}$ on X such that $\widehat{b} \in \Lambda^\perp$ and $\widehat{b}|_Y = b$. From this construction, we see that such a $\widehat{b}$ is unique. Indeed, if $b = 0$ on Y and $\widehat{b} \in \Lambda^\perp$ satisfies $\widehat{b}|_Y = b = 0$, then $\widehat{b}$ must vanish on all of X. This completes the proof of the theorem.

Theorem 1 suggests the following *Approximation Scheme.*

Step 1. Given $f \in C(\Delta)$ and a triangle $\tau \in \Delta$, find $p_\tau \in \pi_k$ such that

$$p_\tau(x_{\alpha,\tau}) = f(x_{\alpha,\tau}) \quad \text{for all } |\alpha| = k.$$

Step 2. Let $s \in S_k^0(\Delta)$ be the spline function given by

$$s|_\tau = p_\tau$$

for each triangle $\tau \in \Delta$. Then find the B-net representation b of s.

Step 3. Find $\widehat{b}$ in accordance with Theorem 1, such that $\widehat{b}|_Y = b|_Y$ and $\widehat{b} \in \Lambda^\perp$. Let g be the spline in $S_k^{r,\mu}(\Delta)$ whose B-net representation b_g agrees with $\widehat{b}$ on X.

We denote by T the linear operator $f \mapsto g$, $f \in C(\Delta)$.

In the sequel, we will denote by a the smallest angle in Δ, and by $\mathrm{Const}_{a,k}$ we mean a constant depending only on a and k, which may vary from situation to situation. We use the notation D_j, $j = 1, 2$, to denote the partial derivative operators with respect to the jth coordinates. Also, the closed star of v, denoted by $\overline{St}(v) =: \overline{St}^1(v)$, is the union of all the triangles attached to v, and the m-star of v, denoted by $\overline{St}^m(v)$, is the union of all triangles that intersect with $\overline{St}^{m-1}(v)$, $m > 1$.

Lemma 3. *The linear operator T satisfies the following conditions*:

(i) *$Tp = p$ for every polynomial $p \in \pi_k$.*
(ii) *If τ is a triangle attached to a vertex u, then*

$$\|(Tf)|_\tau\|_\infty \le \mathrm{Const}_{a,k} \|f|_{N(u)}\|_\infty, \tag{22}$$

where $N(u)$ denotes the star $\overline{St}^{\lfloor r/2\rfloor+2}(u)$.

Proof. The first part of this lemma is a straightforward consequence of the construction of T. The second part will be proved in the next section.

We are now in a position to establish the main result of this paper.

Theorem 2. *If $k \ge 3r + 2$, then there exists a linear operator T from $C^{k+1}(\Delta)$ to $S_k^{r,\mu}(\Delta)$ such that*

$$\|f - Tf\|_\infty \le \mathrm{Const}_{a,k} |\Delta|^{k+1} |f|_{k+1,\infty}, \tag{23}$$

where $|f|_{k+1,\infty} := \sum_{\gamma_1+\gamma_2=k+1} \|D_1^{\gamma_1} D_2^{\gamma_2} f\|_\infty$.

Proof. Let T be the operator described by the above approximation scheme. Let $f \in C^{k+1}(\Delta)$ be given. In order to estimate the error $f - Tf$, we consider

$f(x) - (Tf)(x)$, where x is a point in a triangle τ of Δ. Then there exists a polynomial $p \in \pi_k$ such that $p(x) = f(x)$ and

$$(24) \qquad |p(y) - f(y)| \le \text{Const}_k \, |f|_{k+1,\infty} |\Delta|^{k+1} \quad \text{for all } y \in N(u).$$

By Lemma 3, we deduce from (24) that

$$\begin{aligned} |f(x) - Tf(x)| = |T(f-p)(x)| &\le \|T(f-p)|_\tau\|_\infty \\ &\le \text{Const}_{a,k} \|(f-p)|_{N(u)}\|_\infty \le \text{Const}_{a,k} |f|_{k+1,\infty} |\Delta|^{k+1}. \end{aligned}$$

This estimate is valid for every $x \in \Delta$. Hence, the proof of the theorem is complete.

4. Stable bases with local support

In this section, by using the determining set as described in Section 3, we shall construct a basis for $S_k^{r,\mu}(\Delta)$ which is both stable and local.

To begin with, we establish the following result about the norm estimation of the B-net ordinates of any function in $S_k^{r,\mu}(\Delta)$.

Theorem 3. *Every* $b \in \mathbb{R}^X \cap \Lambda^\perp$ *satisfies*

$$\|b\|_\infty \le \text{Const}_{a,k} \|b|_Y\|_\infty,$$

where Y *is the determining set for* $S_k^{r,\mu}(\Delta)$ *as defined by* (21).

Proof. Let $M := \|b|_Y\|_\infty$. First, we show that, for $n = 0, 1, \ldots, \mu$,

$$(25) \qquad |b(x)| \le \text{Const}_{a,k} \, M, \qquad x \in \bigcup_{u \in V} X_u^n.$$

Let u be any vertex. Among the triangles attached to u, let $\tau = [u, v, w]$ be the one that contains Y_u^n, and let $\tilde{\tau} = [u, v, \tilde{w}]$ be the other triangle attached to the edge $[u, v]$. Since $b \in \Lambda^\perp$, we have

$$b(x_{\alpha,\tilde{\tau}}) = \sum_{|\beta| = \alpha_{\tilde{w}}} \binom{\alpha_{\tilde{w}}}{\beta} \frac{S_u^{\beta_u} S_v^{\beta_v} S_w^{\beta_w}}{S^{\alpha_{\tilde{w}}}} b(x_{(\alpha_u, \alpha_v, 0)+\beta, \tau}),$$

where $\alpha = (\alpha_u, \alpha_v, \alpha_{\tilde{w}}) \in \mathbb{Z}_+^3$ with $|\alpha| = k$ and $\alpha_u \ge k - \mu$. From (16) we see that

$$\left| b(x_{(\alpha_u, \alpha_v, 0)+\beta, \tau}) \right| \le M, \quad \alpha_u \ge k - \mu.$$

Moreover,

$$|S_u^{\beta_u} S_v^{\beta_v} S_w^{\beta_w} / S^{\alpha_{\tilde{w}}}| \le \text{Const}_a.$$

This shows that

$$|b(x_{\alpha,\tilde{\tau}})| \le \text{Const}_{a,k} \, M, \quad \alpha_u \ge k - \mu.$$

Repeating this process, we obtain

$$|b(x)| \le \text{Const}_{a,k} \, M, \quad x \in X_u^n, \; n = 0, 1, \ldots, \mu.$$

Next, we shall prove (25) for $n = \mu + 1, \ldots, 2r$. If u is a singular interior vertex, this can be done by the same argument as before. On the other hand, if u is a nonsingular vertex, we then prove (25) by induction on n as follows. Let n be an integer in $\{\mu+1, \ldots, 2r\}$ and assume that (25) holds for $0, 1, \ldots, n-1$. We wish to prove that (25) also holds for n. For this purpose, we shall employ the smoothness conditions given in Lemma 2. For a triangle τ and $\alpha \in \mathbb{Z}_+^3$ with $|\alpha| = k$, we let $C_{\alpha,\tau}$ be defined as in (5). Let $e = [u, v]$ be an oriented edge attached to u, and let $\tau = [u, v, w]$ and $\tilde{\tau} = [u, v, \tilde{w}]$ be the two triangles with common edge e. It is assumed that the orientation of τ agrees with that of e. By Lemma 2, we have

$$C_{(k-n,n-m,m),\tilde{\tau}} = \sum_{\ell=0}^{m} C_{(k-n,n-\ell,\ell),\tau} \binom{m}{\ell} \frac{S_v^{m-\ell} S_w^{\ell}}{S^m},$$

for $0 \le m \le n \le k$. In order to estimate $C_{\alpha,\tau}$, we introduce $A_{\alpha,\tau}$, as follows. For each triangle $\tau = [u, v, w]$ attached to u and $\alpha = (\alpha_u, \alpha_v, \alpha_w) \in \mathbb{Z}_+^3$ with $|\alpha| = k$, let

$$A_{\alpha,\tau} := C_{\alpha,\tau} \quad \text{for } 0 \le \alpha_v, \alpha_w \le r.$$

Moreover, if $\alpha = (k-n, n-\ell, \ell)$ for some (n, ℓ) with $\mu + 1 \le n \le 2r$ and $2n - 3r - 1 \le \ell \le n - r - 1$, we will consider

$$A_{(k-n,n-\ell,\ell),\tau} := C_{(k-n,n-\ell,\ell),\tau} + \sum_{j=0}^{2n-3r-2} a_{\ell j} C_{(k-n,n-j,j),\tau},$$

where the coefficients $a_{\ell j}$ are to be determined. Fix an integer $j \in \{0, 1, \dots, 2n-3r-2\}$. If $S_v = 0$, we set $a_{\ell j} := 0$ for all $\ell = 2n-3r-1, \dots, n-r-1$; otherwise, let $a_{\ell j}$ be the solutions of the system

$$\sum_{\ell=2n-3r-1}^{n-r-1} a_{\ell j} \binom{m}{\ell} \left(\frac{S_v}{S_w}\right)^{j-\ell} = \binom{m}{j}, \qquad m = n-r, \dots, r,$$

of linear equations. Since the matrix $(\binom{m}{\ell})_{n-r \le m \le r,\, 2n-3r-1 \le \ell \le n-r-1}$ is invertible, there exists a unique solution for $(a_{\ell j})$. The choice of $(a_{\ell j})$ was made in such a way that the equalities

$$A_{(k-n,n-m,m),\tilde{\tau}} = \sum_{\ell=2n-3r-1}^{m} A_{(k-n,n-\ell,\ell),\tau} \binom{m}{\ell} \frac{S_v^{m-\ell} S_w^{\ell}}{S^m} \tag{26}$$

are valid for all (m, n) with $\mu + 1 \le n \le 2r$ and $n - r \le m \le r$. Also, we have

$$\left| a_{\ell j} (S_v/S_w)^{j-\ell} \right| \le \text{Const}_k.$$

Since $\ell \ge 2n - 3r - 1 > j$ and $|S_v/S_w| \le \text{Const}_{a,k}$, we obtain

$$|a_{\ell j}| \le \text{Const}_k |S_v/S_w|^{\ell - j} \le \text{Const}_{a,k}.$$

Next, we define, for convenience,

$$C(x_{\alpha,\tau}) := C_{\alpha,\tau}$$

and

$$A(x_{\alpha,\tau}) := A_{\alpha,\tau} \quad \text{for } x_{\alpha,\tau} \in X_u^n,\ \mu + 1 \le n \le 2r.$$

Then it follows from (26) that

$$\sum_{x \in X_u^n} \lambda(x) A(x) = 0 \quad \forall \lambda \in \Lambda_u^n. \tag{27}$$

Recall from Section 3 that Z_u^n is a subset of X_u^n with $\#Z_u^n = \#\Lambda_u^n$, such that the inequality (19) holds for any subset Z of X_u^n with $\#Z = \#\Lambda_u^n$. Rewrite (27) as

$$\sum_{x\in Z_u^n} \lambda(x)A(x) = -\sum_{x\in X_u^n\setminus Z_u^n} \lambda(x)A(x),$$

and apply Cramer's rule to the above system of linear equations to yield

$$|A(x)| \le \#(X_u^n \setminus Z_u^n) \max_{y\in X_u^n\setminus Z_u^n} |A(y)|, \qquad x \in Z_u^n, \quad \mu+1 \le n \le 2r. \tag{28}$$

Since a is the smallest angle in Δ, the number of triangles attached to the same vertex is bounded from above by a constant depending only on a; hence we have

$$\#(X_u^n \setminus Z_u^n) \le \text{Const}_{a,k}.$$

If $y = x_{\beta,\tau}$ for some β with $\beta_u \ge k-n$ and $y \notin X_u^n$, then from the proof of Theorem 1, we see that $y \in \bigcup_{m=0}^{n-1} (X_{u,e}^m \cup X_{v,e}^m) \cup Y_e$; thus, by the induction hypothesis, we have

$$|b(y)| \le \text{Const}_{a,k}\, M.$$

This together with the construction of $C(y)$ and $A(y)$ implies that

$$|A(y)| \le \text{Const}_{a,k}\, M, \qquad y \in Y_u^n = X_u^n\setminus Z_u^n.$$

Therefore, by (28), we obtain

$$|A(x)| \le \text{Const}_{a,k}\, M, \qquad x \in Z_u^n.$$

Again, by the construction of $C(x)$ and $A(x)$, and by the induction hypothesis, we have

$$|b(x)| \le \text{Const}_{a,k}\, M, \qquad x \in Z_u^n.$$

This establishes (25) for any nonsingular vertex u.

Finally, for $x \in X_e^-$, it is easily seen from the smoothness conditions across the edge e that

$$|b(x)| \le \text{Const}_{a,k}\, M.$$

This completes the proof of the theorem.

Now let us establish an equivalence relation between the norm of a spline function and that of its B-net representation.

Lemma 4. *Let $f \in S_k^0(\Delta)$ and b_f its B-net representation. Then*

$$\|f\|_\infty \le \|b_f\|_\infty \le \text{Const}_k\, \|f\|_\infty. \tag{29}$$

Proof. According to the definition of b_f, we have

$$f(x) = \sum_{|\alpha|=k} b_\alpha B_{\alpha,\tau}(x), \qquad x \in \tau, \tag{30}$$

where $b_\alpha := b_f(x_{\alpha,\tau})$. Since $B_{\alpha,\tau}$ are nonnegative and $\sum_{|\alpha|=k} B_{\alpha,\tau}(x) = 1$ for all $x \in \tau$, it follows that $\|f\|_\infty \le \|b_f\|_\infty$.

In order to prove the second inequality in (29), we consider the standard 2-simplex $\sigma := \{(x_1, x_2) : x_1, x_2 \ge 0;\, x_1 + x_2 \le 1\}$ and a one-to-one affine mapping Q from σ onto τ. Since barycentric coordinates are invariant under affine transforms, we have $B_{\alpha,\sigma}(y) = B_{\alpha,\tau}(Qy)$ for all $y \in \sigma$. Thus, it follows from (30) that

$$f(Qy) = \sum_{|\alpha|=k} b_\alpha B_{\alpha,\sigma}(y).$$

Since $B_{\alpha,\sigma}$, $|\alpha| = k$, constitute a basis of π_k, we have

$$|b_\alpha| \le \mathrm{Const}_k \sup_{y\in\sigma} \{|f(Qy)|\} \le \mathrm{Const}_k \|f\|_\infty.$$

This completes the proof of the lemma.

We are now in a position to describe a procedure for constructing a stable basis of $S_k^{r,\mu}(\Delta)$. For a given point x in Y, it follows from Theorem 1 that there is a unique $B_x \in S_k^{r,\mu}(\Delta)$ whose B-net representation b satisfies

$$b(y) = \begin{cases} 1, & y = x, \\ 0, & y \in Y\backslash\{x\}. \end{cases} \tag{31}$$

Theorem 1 also tells us that $\{B_x : x \in Y\}$ constitutes a basis of $S_k^{r,\mu}(\Delta)$.

Theorem 4. *The basis $\{B_x : x \in Y\}$ of $S_k^{r,\mu}(\Delta)$ is stable in the sense that there are two positive constants K_1 and K_2 depending only on k and a such that*

$$K_1 \sup_{x\in Y} |c_x| \le \left\| \sum_{x\in Y} c_x B_x \right\|_\infty \le K_2 \sup_{x\in Y} |c_x|. \tag{32}$$

This basis is also local in the sense that for any $x \in Y$ there exists a vertex u such that

$$\mathrm{supp} B_x \subseteq \overline{St}^{\lfloor r/2\rfloor+1}(u). \tag{33}$$

Proof. We first prove (32). Let $f = \sum_{x\in Y} c_x B_x$. Then the B-net representation b_f of f satisfies $b_f(x) = c_x$ for all $x \in Y$. By Lemma 4 and Theorem 3, we have

$$\|f\|_\infty \le \|b_f\|_\infty \le \mathrm{Const}_{a,k} \sup_{x\in Y} |c_x|.$$

On the other hand, Lemma 4 implies that

$$\sup_{x\in Y} |c_x| \le \|b_f\|_\infty \le \mathrm{Const}_k \|f\|_\infty.$$

The desired inequality (32) now follows at once from the above estimates.

To prove (33), let $x \in Y$ be arbitrarily chosen. If $x \in Y_\tau$ for some triangle τ, then supp $B_x \subseteq \tau$. Generally, for a given $x \in Y$, there exists a vertex u such that the barycentric coordinate $(\alpha_u, \alpha_v, \alpha_w)$ of x, with respect to any triangle $[u, v, w]$ with u as a vertex, satisfies $\alpha_u \ge \frac{k}{2}$. For two vertices u and v in Δ, we denote by $d(u, v)$ the smallest number of edges among all paths joining u and v. We claim that for any positive integer $m \le 2r + 1 - \mu$, if $d(u, v) \ge m$, then the B-net representation b of B_x vanishes on $\bigcup_{n=0}^{\mu+m-1} X_v^n$. This will be proved by induction on m. If $m = 1$, then for any vertex $v \ne u$, b vanishes on $\bigcup_{n=0}^{\mu} Y_v^n$; hence, by the smoothness conditions around v, we see that b vanishes on $\bigcup_{n=0}^{\mu} X_v^n$. Let $1 < m \le 2r + 1 - \mu$ and assume that our claim has been justified for any positive integer $\ell < m$. We must verify it for m. Suppose that $d(u, v) \ge m$ and $d(v, w) = 1$. Then $d(u, w) \ge m - 1$.

By the induction hypothesis, we see that b vanishes on $\bigcup_{n=0}^{\mu+m-2}(X_v^n \cup X_w^n)$. If $y \in X \setminus Z_v^{\mu+m-1}$ and $\lambda(y) \neq 0$ for some $\lambda \in \Lambda_v^{\mu+m-1}$, then we see from the proof of Theorem 1 that $b(y) = 0$. Hence, b also vanishes on $Z_v^{\mu+m-1}$. This shows that b vanishes on $X_v^{\mu+m-1}$, and therefore completes the induction procedure. If $d(v, u) \geq 2r + 2 - \mu$ and $d(v, w) = 1$, then b vanishes on $\bigcup_{n=0}^{2r} X_v^n$ and $\bigcup_{n=0}^{2r} X_w^n$. Moreover, if one of u and v is an interior vertex, then b vanishes on X_e^-, where e is the oriented edge joining v and w. This shows that b vanishes on the star $\overline{St}(v)$ of the vertex v. Therefore, since $2r + 1 - \mu = 2r + 1 - \lfloor \frac{3r+1}{2} \rfloor = \lfloor \frac{r+2}{2} \rfloor$, we have supp $B_x \subseteq \overline{St}^{\lfloor r/2 \rfloor + 1}(u)$.

It only remains to prove part (ii) of Lemma 3. For $f \in C(\Delta)$, let $s \in S_k^0(\Delta)$ be the spline functions given in the approximation scheme as described in Section 3, and let b be the B-net representation of s. By the construction of Tf, we have

$$Tf(x) = \sum_{y \in Y} b(y) B_y(x), \quad x \in \Delta. \tag{34}$$

Let τ be a triangle of Δ with vertex u and $x \in \tau$. Then $B_y(x) \neq 0$ only if $d(y, u) \leq \lfloor r/2 \rfloor + 2$, or equivalently, $y \in \overline{St}^{\lfloor r/2 \rfloor + 2}(u) = N(u)$. Hence, the number of nonzero terms in (34) is bounded above by $\mathrm{Const}_{a,k}$. Moreover, $\|B_y\|_\infty \leq \mathrm{Const}_{a,k}$ by Theorem 4. Thus, it follows from (34) that

$$|Tf(x)| \leq \mathrm{Const}_{a,k} \max_{y \in N(u) \cap Y} \{|b(y)|\}.$$

By Lemma 4, we may now conclude that

$$\max_{y \in N(u) \cap Y} \{|b(y)|\} \leq \mathrm{Const}_k \|s|_{N(u)}\|_\infty \leq \mathrm{Const}_k \|f|_{N(u)}\|_\infty.$$

Combining the above estimates, we obtain the desired result (22).

Final remarks

1. Recently, de Boor and Jia [3] proved that the order of approximation of $S_k^r(\tilde{\Delta})$ for $k \leq 3r + 1$ and the three direction mesh $\tilde{\Delta}$ is at most k. Hence, $k = 3r + 2$ is the smallest degree for which $S_k^r(\Delta)$ achieves the optimal approximation order of $k + 1$.

2. The main difference between our approach and the previous attempts in [5] and [6] is that the set Z_u^n for Λ_u^n with the property that assertion (28) holds for all $x \in Z_u^n$, $n = \mu + 1, \ldots, 2r$, is obtained by applying (19). Consequently, the dependence of the approximation error on the near-singularity of the triangulation Δ is eliminated. The price to pay is that the supports of the basis functions, as given in Theorem 4, are necessarily larger than those of the vertex splines in [5].

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